

Critical states of transient chaos

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One-dimensional maps exhibiting transient chaos and defined on two preimages of the unit interval $[0,1]$ are investigated. It is shown that such maps have continuously many conditionally invariant measures μ_σ scaling at the fixed point at $x = 0$ as x^σ , but smooth elsewhere. Here σ should be smaller than a critical value σ_c that is related to the spectral properties of the Frobenius-Perron operator. The corresponding natural measures are proven to be entirely concentrated in the fixed point.

I. INTRODUCTION

Transient chaos has attracted an increasing interest in the last decade due to its connection to diffusion [1–4] and chaotic advection [5,6]. Transiently chaotic behavior develops often in a time period preceding the convergence of the trajectories to an attractor, or their escape from the considered region of space as is the case of chaotic scattering. The length of this time-period depends on the starting point of the trajectory and is unlimited, there are trajectories (though with Lebesgue measure zero) that never escape. The behavior of the very long trajectories is governed by the properties of the maximal invariant set, the chaotic repeller and the natural measure on it [7]. This measure is related to the conditionally invariant measure [8], namely the former one is the restriction of the latter one to the repeller accompanied with a normalization to unity there (see for reviews [9,10]).

Transient chaos is much richer in possibilities than the permanent one. Regarding e.g. the frequently studied chaotic systems, the 1D maps, there are rigorous theorems stating that in case of everywhere expanding maps exhibiting permanent chaos there exists a unique absolutely continuous invariant measure. However, this is not any more valid in case of transient chaos for the conditionally invariant measure [11], which in many respects takes over the role of an invariant measure. The main purpose of the present paper is to further investigate this question. It will be shown that one has to distinguish between normal (non-critical) and critical conditionally invariant measures. While the first is typically unique, there are continuously many critical conditionally invariant measures. The latter ones deserve their name since their corresponding natural measure is degenerate. Namely, it is non-zero only on a non-fractal subset of the repeller, on a fixed point in case of 1D maps, we are going to study in the present paper.

The map generated by the function $f(x)$ is assumed to map two subintervals I_0 and I_1 of $[0,1]$ to the whole $[0,1]$ (see Fig. 1). It is monotonically increasing in $I_0 = [0, \hat{x}_0]$ and decreasing in $I_1 = [\hat{x}_1, 0]$, $f(0) = f(1) = 0$

and $f(\hat{x}_0) = f(\hat{x}_1) = 1$. The value of f is undefined in (\hat{x}_0, \hat{x}_1) , and we consider the trajectory being in this interval to escape in the next iteration. We assume f is smooth and hyperbolic ($1 < |f'(x)| < \infty$) in I_0 and I_1 , or we allow singular behavior with infinite slope in \hat{x}_0 , \hat{x}_1 , and $x = 1$.

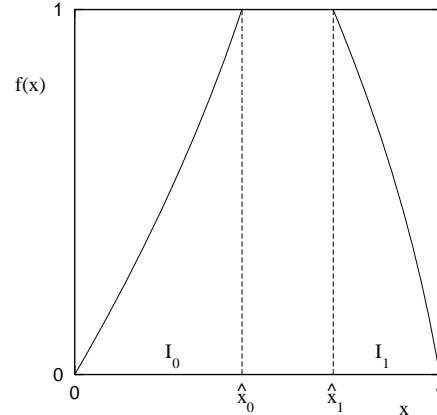


FIG. 1. Schematic plot of the map $f(x)$.

Instead of treating the Frobenius-Perron operator for the density $P^{(k)}(x)$ we deal with the measure $\mu^{(k)}(x) = \mu^{(k)}([0, x])$, where $\mu^{(k)}([x_1, x_2]) = \int_{x_1}^{x_2} P^{(k)}(x) dx$. Note that $\mu^{(k)}(x)$ is a monotonically increasing function. The upper index refers to the discrete time. The equation of time evolution for the measure can be written as

$$\mu^{(k+1)}(x) = T\mu^{(k)}(x) \equiv T_0\mu^{(k)}(x) + T_1\mu^{(k)}(x), \quad (1)$$

where the contributions of the two branches are

$$\begin{aligned} T_0\mu^{(k)}(x) &= \mu^{(k)}(f_0^{-1}(x)), \\ T_1\mu^{(k)}(x) &= \mu^{(k)}(1) - \mu^{(k)}(f_1^{-1}(x)). \end{aligned} \quad (2)$$

$f_0^{-1}(x)$ ($f_1^{-1}(x)$) denotes the lower (upper) branch of the inverse of $f(x)$. Since a portion of the trajectories escapes in every step, normalization is necessary to ensure that the iteration converges to a certain measure [8–10], which is then an eigenfunction of T , namely

$$T\mu(x) = e^{-\kappa}\mu(x). \quad (3)$$

The measure μ is called conditionally invariant measure (the notation μ without an upper index always refers to that in the present paper), and κ is the escape rate. The afore mentioned definition of the natural measure ν yields its connection to μ . Namely, the natural measure [7] of a non-fractal set A ($\mu(A) > 0$) is given by

$$\nu(A) = \lim_{n \rightarrow \infty} \frac{\mu(A \cap f^{-n}[0, 1])}{\mu(f^{-n}[0, 1])}. \quad (4)$$

The paper is organized as follows. The infinity of the coexisting conditionally invariant measures and the general condition for criticality are presented in Section 2. To get a deeper understanding of the conditionally invariant measures we study the spectrum of the Frobenius-Perron operator in Section 3. In Section 4 we show how maps with critical state are connected to each other by singular conjugation, thereby the singular measures are brought into connection with nonsingular ones. In Section 5 it is shown that the natural measures corresponding to critical states are fully concentrated to the fixed point at $x = 0$, which is the main property of criticality. Section 6 is devoted to demonstrate some of the results on examples with further discussion.

II. CONDITIONALLY INVARIANT MEASURES

First we assume that f is nonsingular in I_0 and I_1 , and in the second part of this section we shall study the case when the map may be singular with infinite slope in $x = \hat{x}_0, x = \hat{x}_1$ and/or $x = 1$. We shall see that even in the first case there are continuously many conditionally invariant measures μ_σ that have different power law behavior $\mu_\sigma \sim x^\sigma$ with $\sigma > 0$ at $x = 0$. In order to show this we study what a measure do we get asymptotically when we start with an initial measure $\mu^{(0)}$ that is smooth but scales as $\mu^{(0)}(x) \approx ax^\sigma$ for $x \ll 1$. (The simplest possibility is to choose $\mu^{(0)} = x^\sigma$, which will be used in the numerical calculations. Note that $\sigma = 1$ corresponds to the Lebesgue measure.) The escape rate obtained in the asymptotics shall be denoted by κ_σ . First we study the action of the terms T_i of T on $\mu^{(0)}$ separately (see Eqs. (1),(2)). It can easily be seen that the leading term of $T_0\mu^{(0)}$ is

$$T_0ax^\sigma \approx ae^{-\lambda_0\sigma}x^\sigma \quad \text{if } x \ll 1, \quad (5)$$

where $\lambda_0 = \log(f'(0))$ is the local Liapunov exponent at the fixed point. On the other hand, since T_1 does not take values of $\mu^{(0)}$ from the vicinity of $x = 0$, $T_1\mu^{(0)}$ is smooth with linear behavior

$$T_1\mu^{(0)} \approx bx \quad \text{if } x \ll 1. \quad (6)$$

Therefore $T_0\mu^{(0)}$ dominates in $T_0\mu^{(0)} + T_1\mu^{(0)}$ if $\sigma < 1$, and $T_0\mu^{(0)}$ and $T_1\mu^{(0)}$ both scale linearly near $x = 0$ if

$\sigma = 1$. So the scaling of $\mu^{(0)}$ at $x = 0$ is retained if $\sigma \leq 1$. That is why we expect that the conditionally invariant measure and the corresponding escape rate may depend on σ .

Turning to the asymptotics for large time we rewrite the n -th iterate as

$$T^n\mu^{(0)} = T_0^n\mu^{(0)} + \sum_{k=1}^n T^{k-1}T_1T_0^{n-k}\mu^{(0)}. \quad (7)$$

It can be seen from Eq. (5) that the first term on the r. h. s. of (7) gives a contribution $ae^{-\lambda_0\sigma n}x^\sigma$ for small x . The similar factor in the sum can also be estimated as $T_0^{n-k}\mu^{(0)} \approx ae^{-\lambda_0\sigma(n-k)}x^\sigma$. Acting on this function by T_1 a function is created that is proportional to x in the vicinity of $x = 0$, similarly to (6). Therefore $T_1T_0^{n-k}\mu^{(0)}$ yields an asymptotics $e^{-\kappa_1 k}$ for large k under the action of T^{k-1} . Since for large n at least one of $n - k$ and k is large we obtain

$$T^n\mu^{(0)} \approx ae^{-\lambda_0\sigma n}x^\sigma + \sum_{k=1}^n \mathcal{O}\left(e^{-\lambda_0\sigma(n-k)}e^{-\kappa_1 k}x\right). \quad (8)$$

Consequently, in case $\lambda_0\sigma < \kappa_1$ ($\lambda_0\sigma > \kappa_1$) the first (last) term dominates for large n and small x and $T^n\mu^{(0)}$ behaves asymptotically as $e^{-\lambda_0\sigma n}(e^{-\kappa_1 n})$. That means, there is a critical value $\sigma_c = \kappa_1/\lambda_0$ such that for every $\sigma < \sigma_c$ in the limit $n \rightarrow \infty$ with normalization in each step we obtain a conditionally invariant measure μ_σ with leading term proportional to x^σ at $x = 0$, while in case $\sigma > \sigma_c$ we obtain μ_1 . It is easy to see applying T on μ_1 that $\kappa_1 < \lambda_0$, i. e. $\sigma_c < 1$. The corresponding escape rates are

$$\kappa_\sigma = \lambda_0\sigma \quad \text{if } \sigma < \sigma_c, \quad (9)$$

$$\kappa_\sigma = \kappa_1 \quad \text{if } \sigma > \sigma_c. \quad (10)$$

So the escape rate in case $\sigma < \sigma_c$ is determined by the slope taken at the fixed point $x = 0$. We consider the system to be critical with respect to μ_σ if $\sigma < \sigma_c$ since the density of the corresponding natural measure is a Dirac delta function located at the origin, as will be shown in Section 4. Deeper understanding of Eqs. (9), (10) shall be reached in the next section by studying the spectrum of T .

In the second part of this section we allow singularities of f at the maximum points $\hat{x}_i = f_i^{-1}(1)$ with $i = 0, 1$ and/or at $x = 1$. Namely, the inverse branches behave as

$$f_i^{-1}(x) \approx \hat{x}_i + B_i(1-x)^\psi \quad \text{if } 1-x \ll 1, \quad (11)$$

$$f_1^{-1}(x) \approx 1 - Cx^\omega \quad \text{if } x \ll 1, \quad (12)$$

where $\psi \geq 1$ and $\omega \geq 1$. If $\psi > 1$ ($\omega > 1$) the map has infinite slope at the maximum points \hat{x}_0, \hat{x}_1 (at the point $x = 1$). Studying the effect of one iteration on a monotonic function $\chi(x)$ that is smooth in $(0, 1)$ and obeys scaling $\chi(x) \approx ax^\sigma$ at $x = 0$ we see that Eq. (9)

remains valid. $T_0\chi(x)$ and $T_1\chi(x)$ have leading terms with exponent ψ at $x = 1$,

$$T_i\chi(x) \approx T_i\chi(1) - b_i(1-x)^\psi, \quad i = 0, 1 \quad \text{if } 1-x \ll 1. \quad (13)$$

If $\chi(x)$ scales as $\chi(x) \approx \chi(1) - b(1-x)^\psi$ near $x = 1$ then acting by T_1 on it the result scales as

$$T_1\chi(x) \approx cx^\beta \quad \text{with } \beta = \psi\omega \quad \text{if } x \ll 1. \quad (14)$$

Consequently, starting with a $\chi(x)$ that satisfies

$$\chi(x) \approx ax^\sigma \quad \text{if } x \ll 1, \quad (15)$$

$$\chi(x) \approx \chi(1) - b(1-x)^\psi \quad \text{if } 1-x \ll 1 \quad (16)$$

$T\chi$ retains these properties if $\sigma \leq \beta$.

We start with a $\mu^{(0)}$ in a class given by Eqs. (15), (16) and investigate Eq. (7) similarly to the case of maps non-singular in I_0, I_1 , which maps correspond to the case $\beta = 1$. By Eq. (5) we obtain again that $T_0^n\mu^{(0)} \approx ae^{-\lambda_0\sigma n}x^\sigma$ and $T_0^{n-k}\mu^{(0)} \approx ae^{-\lambda_0\sigma(n-k)}x^\sigma$ near $x = 0$. According to Eqs. (13) and (14) $T_1T_0^{n-k}\mu^{(0)}$ belongs to the class of functions defined by Eqs. (15) (16) with $\sigma = \beta$. Its iterates by T^{k-1} decay proportionally to $e^{-\kappa_\beta k}$ for large k . Since for large n either k or $n-k$ is large, we finally estimate $T^n\mu^{(0)}$ using Eq. (7) as

$$T^n\mu^{(0)} \approx ae^{-\lambda_0\sigma n}x^\sigma + \sum_{k=1}^n \mathcal{O}\left(e^{-\lambda_0\sigma(n-k)}e^{-\kappa_\beta k}x\right). \quad (17)$$

That means, the border value of σ is now $\sigma_c = \kappa_\beta/\lambda_0$. In case $\sigma < \sigma_c$ we obtain a conditionally invariant measure μ_σ belonging to the class of functions given by Eqs. (15), (16), while in case $\sigma > \sigma_c$ we obtain the conditionally invariant measure μ_β , which belongs to the class given by (15), (16) with $\sigma = \beta$. Applying T on μ_β one can easily see that $\kappa_\beta < \lambda_0$, i. e. $\sigma_c < \beta$. The corresponding escape rates are

$$\kappa_\sigma = \lambda_0\sigma \quad \text{if } \sigma < \sigma_c, \quad (18)$$

$$\kappa_\sigma = \kappa_\beta \quad \text{if } \sigma > \sigma_c. \quad (19)$$

Again the escape rate in case $\sigma < \sigma_c$ is determined alone by the slope of the map at $x = 0$ and the measure belonging to such a σ represents a critical state. Let us emphasize that while there is a continuum infinity of critical conditionally invariant measures the noncritical one is unique.

It seems to be impossible to determine the full basin of attraction of the conditionally invariant measures. In the class of functions that are monotonic and smooth in $(0, 1)$ those belong to the basin of attraction of μ_σ with $\sigma < \sigma_c$ that

- scale as ax^σ at $x = 0$ and not slower than $\mu^{(0)}(1) - b(1-x)^{\sigma/\omega}$ at $x = 1$,

- or scale faster than ax^σ at $x = 0$ and scale as $\mu^{(0)}(1) - b(1-x)^{\sigma/\omega}$ at $x = 1$.

The basin of attraction of μ_β consists of the functions that scale faster than ax_c^σ at $x = 0$ and faster than $\mu^{(0)}(1) - b(1-x)^{\sigma_c/\omega}$ at $x = 1$.

Note that the possible singular behavior of the noncritical conditionally invariant measure is determined completely by the map. The behavior of critical conditionally invariant measures on the right hand side is also determined by the map. By this reason we classify these critical measures by the behavior near $x = 0$. Their leading term at $x = 0$ is analytic when σ is integer. The number of such measures is $[\sigma_c]$, where $[\]$ denotes integer part. If $\psi = \omega = 1$ then $[\sigma_c] = 0$.

III. EIGENVALUE SPECTRUM

The conditionally invariant measures obtained in the previous section are particular eigenfunctions of the operator T (see Eqs. (1,2)), namely, they are monotonous (positive definite) functions. To get further insight into the appearance of the upper value σ_c of the parameter σ specifying the critical measures we study more general eigenfunctions of the operator T . We allow that f may have singularity at $x = \hat{x}_0$, $x = \hat{x}_1$ and/or in $x = 1$, as in the second part of the previous section. However, for sake of simplicity here we also assume that the inverse branches of the map are analytic, so ψ and ω in Eqs. (11), (12) (and thereby β in (14)) are integers. We also assume, as it is typical, that there is a discrete spectrum of the Frobenius-Perron operator in the space of analytic functions. This has been proved for certain one-parameter families of maps [12–14,10].

We shall see that for any value of σ an expansion in terms of the basis functions $T_0x^{\sigma+n}$ and $x^{\beta+n}$ with $n = 0, 1, \dots$ is convenient for the search of eigenfunctions. Therefore we start from the form

$$\phi = \sum_{n=0}^{N(\sigma)-1} c_n T_0 x^{\sigma+n} + \sum_{n=0}^{\infty} d_n x^{\beta+n}, \quad (20)$$

where $N(\sigma) = \beta - \sigma$ if σ is integer, and $N(\sigma) = \infty$ otherwise. The limitation by $N(\sigma)$ is necessary if σ is integer, since $T_0x^{\sigma+n}$ can be expanded on the basis functions $x^{\beta+l}$ if $\sigma + n$ is an integer greater or equal to β . Note that

$$T_0x^{\sigma+n} = \sum_{m=0}^{\infty} g_{mn}x^{\sigma+m}. \quad (21)$$

It also follows that

$$g_{mn} = e^{-\lambda_0(\sigma+m)} \quad \text{if } m = n, \quad (22)$$

$$g_{mn} = 0 \quad \text{if } m < n, \quad (23)$$

and the basis functions in the first sum of Eq. (20) are transformed by T_0 as

$$T_0 T_0 x^{\sigma+n} = \sum_{m=0}^{\infty} g_{mn} T_0 x^{\sigma+m} . \quad (24)$$

The transformation by T_1 can be obtained similarly to Eq. (14),

$$T_1 T_0 x^{\sigma+n} = \sum_{m=0}^{\infty} H_{mn} x^{\beta+m} . \quad (25)$$

Clearly, the basis functions in the second sum of Eq. (20) are transformed by T in the way

$$T x^{\beta+n} = \sum_{m=0}^{\infty} Q_{mn} x^{\beta+m} . \quad (26)$$

As seen from Eqs. (24), (25) and (26) the iteration of the vectors \mathbf{c}, \mathbf{d} formed from the expansion coefficients in Eq. (20) under the action of T can be described as

$$T \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{H} & \mathbf{Q} \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} , \quad (27)$$

where \mathbf{G} is a matrix constructed from the coefficients g_{mn} but with truncation to size $N(\sigma) \times N(\sigma)$ if σ is integer.

In case $\mathbf{c} = 0$ only \mathbf{Q} is in effect, so the eigenvalue problem yields the eigenvalues $\Lambda_{\beta,n}$ of \mathbf{Q} and corresponding eigenfunctions $\phi_{\beta,n}$, whose expansion starts with x^β . On the other hand, since \mathbf{G} is a triangular matrix its leading eigenvalue is $e^{-\lambda_0 \sigma}$ (see Eqs. (22), (23)) that belongs to an eigenvector denoted by \mathbf{c}_σ . Finally, an eigenfunction ϕ_σ of T with an x^σ scaling at $x = 0$ can be obtained in the form of (20) with eigenvalue Λ_σ and coefficients

$$\mathbf{c} = \mathbf{c}_\sigma , \quad \mathbf{d} = (\mathbf{g}_{00} - \mathbf{Q})^{-1} \mathbf{H} \mathbf{c}_\sigma , \quad (28)$$

where $\Lambda_\sigma = \mathbf{g}_{00} = e^{-\lambda_0 \sigma}$, except the special cases when σ is an integer greater or equal to β or Λ_σ coincides with an eigenvalue $\Lambda_{\beta,n}$ of \mathbf{Q} . Thereby we have obtained besides the assumed discrete spectrum of analytic eigenfunctions an almost continuous spectrum of T . Most of these eigenfunctions have nonanalyticity due to noninteger value of σ .

The next important question is, how the conditionally invariant measures can be selected from these eigenfunctions. A measure should be nonnegative for any set, so the condition is that $\phi(x)$ should be monotonous. This is ensured when it can be generated starting from a monotonous $\phi^{(0)}(x)$ as the limit of iteration $T^n \phi^{(0)}(x)$, normalizing it in each step. We can study this condition using the above results. After one iteration of a general monotonic $\phi^{(0)}$ with $\mathcal{O}(x^\sigma)$ scaling at $x = 0$ the iterate $T \phi^{(0)}$ is a linear combination of the basis functions $T_0 x^{\sigma+n}$ and $x^{\beta+n}$ with $n \geq 0$. Therefore the limit of infinite iterations yields the eigenfunction that has the largest eigenvalue among the eigenfunctions that can be expanded on this basis. That one is the eigenfunction ϕ_σ if $\Lambda_\sigma = e^{-\lambda_0 \sigma} > \Lambda_{\beta,0}$, i. e. $\sigma < -\log(\Lambda_{\beta,0})/\lambda_0$. On the other hand, when starting with an initial $\phi^{(0)}$

with $\mathcal{O}(x^\beta)$ scaling at $x = 0$ we do not get terms with any smaller exponent, thereby we obtain the eigenfunction $\phi_{\beta,0}$. Therefore the correspondence between these eigenfunctions and the conditionally invariant measures and between the eigenvalues and escape rates can be described as

$$\phi_\sigma = \mu_\sigma , \quad \Lambda_\sigma = e^{-\kappa_\sigma} \quad \text{if } \sigma < \sigma_c = \kappa_\beta / \lambda_0 , \quad (29)$$

$$\phi_{\beta,0} = \mu_\beta , \quad \Lambda_{\beta,0} = e^{-\kappa_\beta} . \quad (30)$$

The spectrum of the Frobenius-Perron operator allowing singular eigenfunctions has been studied for piecewise linear maps by MacKernan and Nicolis [15]. Except the tent map, they considered eigenfunctions singular at internal points of the interval. They pointed out the existence of the continuous parts of the spectrum, but did not raise the question of the possible monotonous property of the eigenfunctions for a region of eigenvalues, which has been our main concern here.

Finally we note that singular eigenfunctions of the generalized Frobenius-Perron operator have been of importance in the thermodynamic formalism to describe phase transition like phenomena, where the "temperature" has played the role of the control parameter [16,17].

IV. CONJUGATION

It can be seen that all critical systems can be brought to the same form by the application of smooth conjugation. For this purpose a conjugation function u has to be introduced, which is smooth everywhere except in $x = 0$ and $x = 1$, where it may be singular. These singularities can be characterized by the exponents η and α :

$$u(x) \approx x^\eta \quad \text{if } x \ll 1 , \quad (31)$$

$$u(x) \approx 1 - (1 - x)^\alpha \quad \text{if } 1 - x \ll 1 . \quad (32)$$

By definition, the conjugation transforms the map and the measure in the following way:

$$\tilde{f}_i^{-1}(x) = u \left(f_i^{-1} (u^{-1}(x)) \right) , \quad (33)$$

$$\tilde{\mu}(x) = \mu (u^{-1}(x)) . \quad (34)$$

To see how the conjugation changes the exponents important from the point of view of criticality, transformation (34) has to be applied to the conditionally invariant measure, and transformation (33) to the branches of the inverse map. The conditionally invariant measure μ is in the class of functions given by Eqs. (15,16) and the branches of the inverse map are described in Eqs. (11,12). The conjugation results in the following transformation rules for the characterizing exponents:

$$\tilde{\psi} = \frac{\psi}{\alpha} , \quad (35)$$

$$\tilde{\omega} = \alpha \frac{\omega}{\eta} , \quad (36)$$

$$\tilde{\lambda}_0 = \lambda_0 \eta, \quad (37)$$

$$\tilde{\sigma} = \frac{\sigma}{\eta}. \quad (38)$$

From these transformation rules it is clearly seen that by the application of the appropriately chosen u any two of the three quantities ψ , ω and σ can be set to unity. This means all critical systems can be brought to the same form, which shows that criticality is the same, regardless it is caused by the singular measure or the singularity of the map in $x = \hat{x}_0$, $x = \hat{x}_1$ and $x = 1$.

It is worth noting that any conditionally invariant measure can be chosen as conjugating function. The conjugation in this case results in the equivalent map, which has the Lebesgue measure as a conditionally invariant one. Such maps will be called Lebesgue maps in the following. The equivalent Lebesgue map will be denoted by $\tilde{f}^{(\sigma)}$, if the conditionally invariant measure chosen for the conjugation is μ_σ , i.e. the measure decaying at $x = 0$ with the exponent σ . Naturally, the Lebesgue measure is not singular in $x = 0$, so $\tilde{\sigma} = 1$. Moreover, since the conditionally invariant measure is asymptotically proportional to $(1-x)^\psi$ in $x = 1$, the conjugation sets the exponent $\tilde{\psi}$ to unity, too, so after the conjugation both $\tilde{\sigma}$ and $\tilde{\psi}$ are equal to unity. However, if criticality is in existence, i.e. $\sigma < \sigma_c$, then $\tilde{\omega}$ will be greater than one, i.e. the equivalent Lebesgue map is singular in $x = 1$. It is obvious that the map has as many equivalent Lebesgue maps as conditionally invariant measures.

V. PROPERTIES OF THE NATURAL MEASURE

During the investigations of the piecewise parabolic map it was found that the natural measure of the fixed point at $x = 0$ is positive when the Lebesgue-measure as initial measure was iterated [18]. Numerical results suggested, and later it was supported by analytical considerations that this measure is not only positive but is equal to unity [18,19]. We shall prove here that this phenomenon is quite a general property: for any map with critical conditionally invariant measure the natural measure of the fixed point at $x = 0$ is equal to unity.

Since f_0^{-1} has a finite slope e^{λ_0} at $x = 0$,

$$C_n(x) = \frac{f_0^{-n}(x)}{e^{-\lambda_0 n x}} \xrightarrow{n \rightarrow \infty} C_\infty(x), \quad (39)$$

where $0 < C_\infty(x) < \infty$. Furthermore, the critical conditionally invariant measure μ is asymptotically proportional to x^σ , that is

$$m(x) = \frac{\mu(x)}{x^\sigma} \xrightarrow{x \rightarrow 0} M, \quad (40)$$

where $\sigma < \sigma_c$ and M is finite and positive. We introduce the following notation for the set of the preimages of the unit interval $I = I_0^{(0)} = [0, 1]$. The first two preimage

intervals are $I_0^{(1)} = f_0^{-1}(I)$ and $I_1^{(1)} = f_1^{-1}(I)$. Similarly, the $(n+1)$ -th preimages can be generated from the n -th ones as $I_i^{(n+1)} = f_0^{-1}(I_i^{(n)})$ and $I_{2^n+i}^{(n+1)} = f_1^{-1}(I_i^{(n)})$. The set of all the n -th preimages of I is denoted by $I^{(n)} = \bigcup_{i=0}^{2^n-1} I_i^{(n)}$. We want to determine the natural invariant measure of a single point, the fixed point located at $x = 0$, a series of intervals containing this point must be found, whose limit is the fixed point itself. The natural measure of the fixed point is equal to the limit of the series of the natural measures of these intervals. The series of the leftmost intervals of the k -th interval sets is an appropriate and convenient choice, so the natural measure of $x = 0$ is

$$\nu(\{0\}) = \lim_{k \rightarrow \infty} \nu(I_0^{(k)}) = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\mu(I_0^{(k)} \cap I^{(n)})}{\mu(I^{(n)})}. \quad (41)$$

Since $\mu(I_0^{(k)} \cap I^{(n)}) > \mu(I_0^{(n)})$, $k < n$, we estimate the natural measure from below by keeping only the leftmost interval. From the criticality and Equations (40) and (39) follows

$$\begin{aligned} \nu(\{0\}) &\geq \lim_{n \rightarrow \infty} \frac{\mu(f_0^{-n}(1))}{\mu(I^{(n)})} \\ &= \lim_{n \rightarrow \infty} \frac{m(C_n(1)e^{-\lambda_0 n}) \cdot C_n(1)^\sigma e^{-\lambda_0 n \sigma}}{e^{-\kappa n}} \\ &= M \cdot C_\infty(1)^\sigma > 0, \end{aligned} \quad (42)$$

so the positive natural measure of the fixed point is proven.

It can also be shown that this measure is equal to unity. For this purpose the features of the conjugation to an equivalent Lebesgue map have to be used. Let us choose as the conjugating function the non-critical conditionally invariant measure μ_β , where $\beta = \psi\omega$. The conjugation results in the map $\tilde{f}^{(\beta)}$, which is characterized by exponents $\tilde{\psi} = 1$ and $\tilde{\omega} = \beta$. The index (β) in the following will be omitted. Any μ_σ conditionally invariant measure transforms into $\tilde{\mu}_{\tilde{\sigma}}$, where $\tilde{\sigma} = \sigma/\beta$. Neither μ_β nor its conjugated pair, the $\tilde{\mu}_1$ Lebesgue measure are critical, so $\tilde{\sigma} < \tilde{\sigma}_c < 1$ must hold for any critical $\tilde{\mu}_{\tilde{\sigma}}$ measure. Since $\tilde{\mu}_1$ is Lebesgue measure, the $\tilde{\ell}(\tilde{I}^{(n)})$ total length of the n -th preimage set of the unit interval I is equal to its measure with respect to $\tilde{\mu}_1$. This fact makes the exact determination of $\tilde{\ell}(\tilde{I}^{(n)})$ possible. Since $\mu(f^{-n}(A)) = T^n \mu(A) = e^{-\kappa n} \mu(A)$ for any set $A \subseteq I$ and μ conditionally invariant measure

$$\tilde{\ell}(\tilde{I}^{(n)}) = \tilde{\mu}_1(\tilde{f}^{-n}(\tilde{I})) = e^{-\tilde{\kappa}_1 n} = e^{-\kappa_\beta n} \quad (43)$$

holds. Now we can calculate the natural measure concentrated in the fixed point at $x = 0$ for $\tilde{\mu}_{\tilde{\sigma}}$, $\sigma < \sigma_c$ critical conditionally invariant measures. For this purpose

we can use Eq. (41). Since, provided that $k < n$, $\tilde{I}_0^{(k)} = [0, \tilde{f}_0^{-k}(1)]$ and $\tilde{I}^{(n)} = [0, \tilde{f}_0^{-k}(1)] \cap \tilde{I}^{(n)} \cup [\tilde{f}_0^{-k}(1), 1] \cap \tilde{I}^{(n)}$, we can write that

$$\begin{aligned} \tilde{\nu}_{\tilde{\sigma}}(\{0\}) &= \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\tilde{\mu}_{\tilde{\sigma}}(\tilde{I}_0^{(k)} \cap \tilde{I}^{(n)})}{\tilde{\mu}_{\tilde{\sigma}}(\tilde{I}^{(n)})} \\ &= \lim_{k \rightarrow \infty} \frac{1}{1 + \lim_{n \rightarrow \infty} \frac{\tilde{\mu}_{\tilde{\sigma}}([\tilde{f}_0^{-k}(1), 1] \cap \tilde{I}^{(n)})}{\tilde{\mu}_{\tilde{\sigma}}([0, \tilde{f}_0^{-k}(1)] \cap \tilde{I}^{(n)})}}. \end{aligned} \quad (44)$$

The measure $\tilde{\mu}_{\tilde{\sigma}}([0, \tilde{f}_0^{-k}(1)] \cap \tilde{I}^{(n)})$ can be treated similarly as $\mu(f_0^{-n}(1))$ in Equation (42):

$$\begin{aligned} \tilde{\mu}_{\tilde{\sigma}}([0, \tilde{f}_0^{-k}(1)] \cap \tilde{I}^{(n)}) &\geq \tilde{\mu}_{\tilde{\sigma}}(\tilde{I}_0^{(n)}) \\ &= m \left(\tilde{C}_n(1) e^{-\tilde{\lambda}_0 n} \right) \tilde{C}_n(1)^{\tilde{\sigma}} e^{-\tilde{\lambda}_0 \tilde{\sigma} n}. \end{aligned} \quad (45)$$

The expression $\tilde{\mu}_{\tilde{\sigma}}([\tilde{f}_0^{-k}(1), 1] \cap \tilde{I}^{(n)})$ is the measure of an interval set located in $[\tilde{f}_0^{-k}(1), 1]$ with total length not greater than $e^{-\tilde{\kappa}_1 n}$, which is the length of the n -th preimage interval set. This measure is not greater than the maximum of the measure of such interval sets. Since for any fixed value k there exist a $\tilde{\mu}'_{\tilde{\sigma}, \max}(k)$ finite upper bound of the derivative of the conditionally invariant measure in $[\tilde{f}_0^{-k}(1), 1]$,

$$\tilde{\mu}_{\tilde{\sigma}}([\tilde{f}_0^{-k}(1), 1] \cap \tilde{I}^{(n)}) \leq \tilde{\mu}'_{\tilde{\sigma}, \max}(k) e^{-\tilde{\kappa}_1 n}. \quad (46)$$

Using inequalities (45), (46) and that $\tilde{\lambda}_0 \tilde{\sigma} < \tilde{\kappa}_1$ due to the criticality, the limit of the fraction in the denominator of the right hand side of Eq. (44) is equal to zero, so

$$\tilde{\nu}_{\tilde{\sigma}}(\{0\}) = 1 \quad (47)$$

whenever $\tilde{\mu}_{\tilde{\sigma}}$ is a critical conditionally invariant measure. Now we prove that not only the conjugated natural measure, but the original one is concentrated in the fixed point, too. We have already seen that for any fixed k

$$\lim_{n \rightarrow \infty} \frac{\tilde{\mu}_{\tilde{\sigma}}(\tilde{I}_0^{(k)} \cap \tilde{I}^{(n)})}{\tilde{\mu}_{\tilde{\sigma}}(\tilde{I}^{(n)})} = 1. \quad (48)$$

Since the conjugation does not change the measure of any single interval, i.e. $\mu_{\sigma}(I_i^{(n)}) = \tilde{\mu}_{\tilde{\sigma}}(\tilde{I}_i^{(n)})$, the same equation applies for the non-conjugated map, which means

$$\nu_{\sigma}(\{0\}) = 1 \quad (49)$$

for any critical measures.

By using this critical measure as conjugating function one can get the equivalent Lebesgue map $\tilde{f}^{(\sigma)}$ where the Lebesgue measure represents the critical state. The density of its corresponding natural measure is $\delta(x + 0)$. In this Lebesgue map the density of the measure of the coarse grained repeller $I^{(n)}$ is given by the n -th iterate of $P^{(0)}(x) = 1$ by the adjoint of the Frobenius-Perron equation:

$$L^+ g = \begin{cases} g(\tilde{f}^{(\sigma)}(x)) & \text{if } \tilde{f}^{(\sigma)}(x) \in [0, 1], \\ 0 & \text{if } \tilde{f}^{(\sigma)}(x) \notin [0, 1]. \end{cases} \quad (50)$$

This equation has $\delta(x + 0)$ as eigenfunction with eigenvalue $e^{\kappa \sigma} = \left. \frac{d}{dx} \tilde{f}^{(\sigma)}(x) \right|_{x=0}$. Our result (47) amounts to proving that $L^{+n} P^{(0)}(x)$ converges to $\delta(x + 0)$ when $n \rightarrow \infty$. This convergence property has been assumed previously supported by numerical calculations and also some of its consequences have been exploited [18,19,11].

From Equation (49) follows that $\lambda_0 = \lambda$, where λ is the average Liapunov exponent, and the Kolmogorov-Sinai entropy K is zero for the natural measure in the critical case. The equations

$$\kappa = \sigma \lambda, \quad K = 0 \quad (51)$$

valid for the critical states are the counterparts of the generalized Pesin relation

$$\kappa = \lambda - K \quad (52)$$

valid for noncritical states, obtained by Kantz and Grassberger [7,20].

Finally we note, that the map f can be considered to be the reduced map of a translationally invariant map of the real axis [21–23,2,4]. Then the diffusion coefficient can be written as an average over the natural measure of the reduced map [11]. This in case of critical state obviously results in a zero diffusion coefficient. In the noncritical state there are important connections between the diffusion and the formula (52) [1,24,25].

VI. EXAMPLES

In this section we demonstrate the properties we have found along with further discussion. As an example consider the map whose inverse branches are

$$\begin{aligned} f_0^{-1}(x) &= \frac{1+d}{2R}x - \frac{d}{4R^2}x^2, \\ f_1^{-1}(x) &= 1 - \frac{1-d}{2R}x - \frac{d}{4R^2}x^2, \end{aligned} \quad (53)$$

where $R > 1$ and $-1 < d \leq 1$ must hold. The case $d = 0$ corresponds to the case of the tent map and the eigenvalue $\Lambda_{\sigma} = (2R)^{-\sigma}$ has been already given by [15]. Eq. (29) for $d = 0$ shows in which region one can connect this eigenvalue with the escape rate. The map is conjugated to the symmetric piecewise parabolic map [26,18,19,11]. For the sake of simplicity we limit our discussion to non-negative values of d . Substituting the inverse branches into the Frobenius-Perron equation, one can immediately see that the Lebesgue measure is a conditionally invariant measure with the escape rate $\kappa_1 = \log R$, independently of the value of d . Similarly, the exponent ψ is equal to unity for any $0 \leq d \leq 1$. However, ω and consequently $\beta = \psi \omega$ have two possible values depending on d . This

makes it sensible to analyse this map in two parts, according to the value of β . Let us start with $0 \leq d < 1$, when $\beta = \omega = 1$. The value of $\kappa_\beta = \kappa_1 = \log R$ is exactly known, therefore

$$\sigma_c = \frac{\kappa_\beta}{\lambda_0} = \frac{\log R}{\log R + \log \frac{2}{1+d}}. \quad (54)$$

Numerical results for κ_σ fit to Eqs. (9), (10), as seen in Fig. 2. Fig. 3 shows some of the numerically obtained conditionally invariant densities.

In the case $d = 1$ obviously $\beta = \omega = 2$. Then κ_β is not known exactly, but it can be determined numerically. Numerical calculation for $R = 1.5$ gave $\kappa_\beta \approx 0.60$ and $\sigma_c = \kappa_\beta / \log R \approx 1.48$. Accordingly to the results of Section 2 conditionally invariant measures were found for $\sigma < \sigma_c$ and values of κ_σ fit to Eqs. (18), (19) (see Figs. 2 and 3). However, critical slowing down of convergence is seen near σ_c .

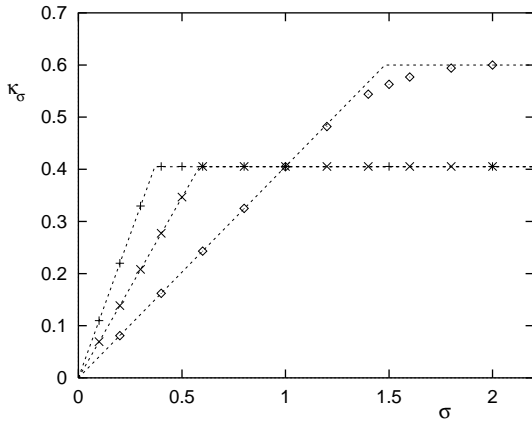


FIG. 2. Numerical results for κ_σ in case of the map (53) with $R = 1.5$ and $d = 0$ (+), $d = 0.5$ (\times), $d = 1$ (\diamond); and the theoretical results (9), (10), (18), (19) (dashed lines) for the same values of d .

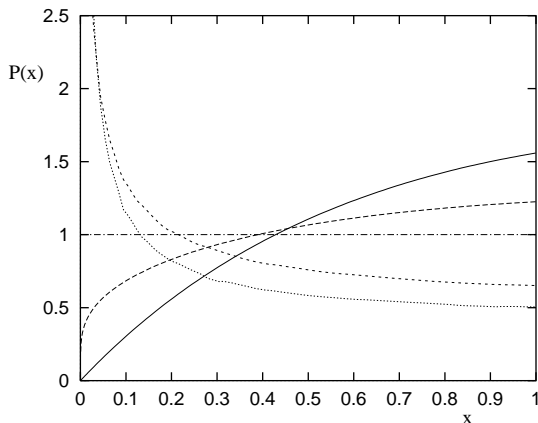


FIG. 3. The densities of the conditionally invariant measures of the map (53) with $d = 0, \sigma = 0.2$ (dotted line); $d = 0.5, \sigma = 0.4$ (short dashes); $d = 1, \sigma = 1.2$ (long dashes); $d = 1, \sigma = 2$ (solid line); and the constant density of the Lebesgue measure obtained numerically for $\sigma = 1$ and all the three values of d (dashed dotted line).

Another map was constructed for that $\psi = 1$ and $\beta = \omega = 4$. Its inverse branches are

$$\begin{aligned} f_0^{-1}(x) &= \frac{x}{R} - \frac{x^4}{Q}, \\ f_1^{-1}(x) &= 1 - \frac{x^4}{Q}, \end{aligned} \quad (55)$$

where $R > 1$ and $Q \geq 4R$. In the numerical calculations $R = 1.25$ and $Q = 40$ was used. The Lebesgue measure is again one of the conditionally invariant measures with escape rate $\kappa_1 = \log R$. From the numerical value $\kappa_\beta \approx 0.730$ follows that $\sigma_c = \kappa_\beta / \log R \approx 3.27$. Numerical values of κ_σ are compared to the theoretical values in Fig. 4. Presence of the conditionally invariant measure that is smooth at least in the inside of $[0, 1]$ was checked numerically at several values of σ with $\sigma < \sigma_c$ and at $\sigma = \beta$. Among them the ones with integer σ have analytic leading term at $x = 0$. The densities of the latter ones together with a singular one are seen in Fig. 5.

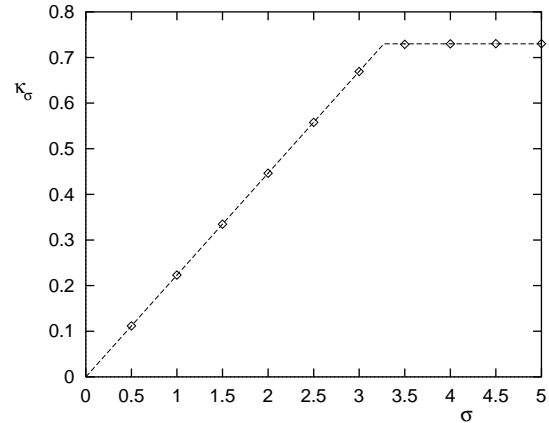


FIG. 4. Numerical (\diamond) and theoretical (dashed line) results for κ_σ in case of the map (55).

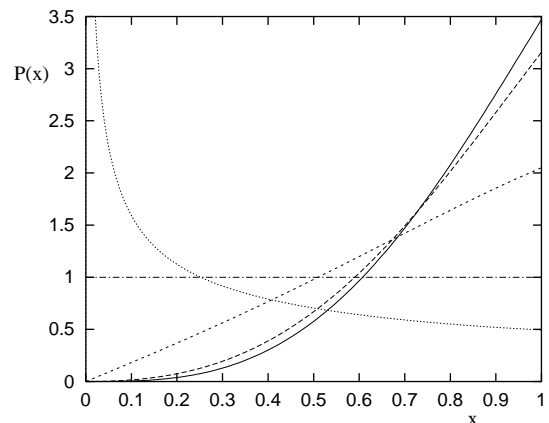


FIG. 5. The conditionally invariant densities of the map (55) with $\sigma = 0.5$ (dotted line), $\sigma = 1$ (dashed dotted line), $\sigma = 2$ (short dashes), $\sigma = 3$ (long dashes), and $\sigma = 4$ (solid line).

ACKNOWLEDGMENTS

This work has been supported in part by the Hungarian National Scientific Research Foundation under Grant Nos. OTKA T017493 and OTKA F17166.

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